Geometric construction of square root of a positive integer

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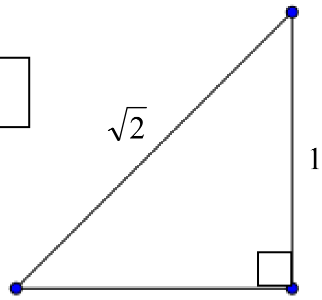
The **aim** of this article is to construct geometrically some surds of the form $\sqrt{n}$ by Pythagoras Theorem where n is a positive integers.

(1) $\sqrt{2}$

The geometric construction of the irrational number $\sqrt{2}$ is classical and is pretty easy. Draw a triangle with two sides of length 1 adjacent to a right angle.Using Pythagoras Theorem, the hypotenuseis then of length $\sqrt{1^2+1^2}=\sqrt{2}$.

Figure 1



(2) $\sqrt{3}$

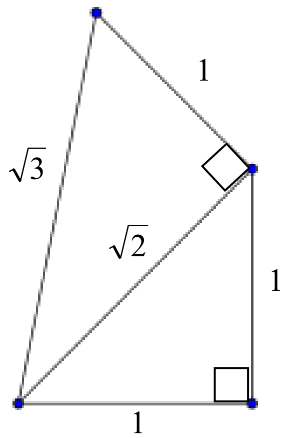
(a) Two right-angled triangles solution

As in the diagram, by Pythagoras Theorem:

$$\sqrt{2}=\sqrt{1^1+1^2}$$

$$\sqrt{3}=\sqrt{\left(\sqrt{2}\right)^2+1^2}$$

Figure 2



(b) One right-angled triangle solution

(1) Construct a vertical line AB of length 1.

(2) Construct a horizontal line BC.

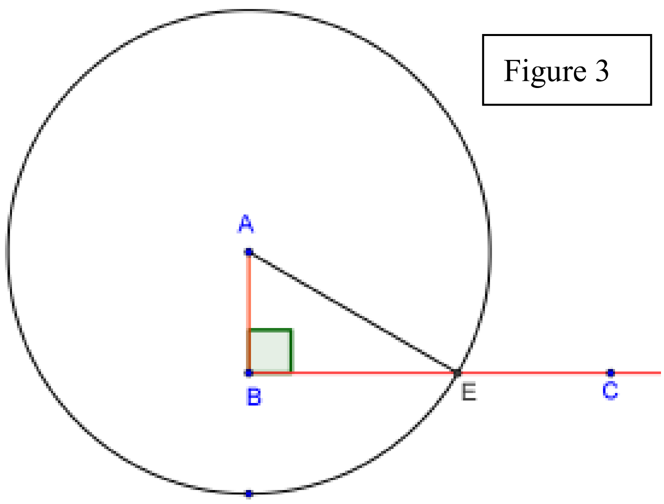
$\therefore\ \angle ABC = 90^o$

(3) With centre A, draw a circle with radius 2.

(4) The circle cuts BC at E.

(5) Join AE.

Figure 3



Then by Pythagoras Theorem:

$$AE = \sqrt{2^2 - 1^2} = \sqrt{4-1} = \sqrt{3}$$

We call this "Subtraction method" , as there is "– " sign in the "$\sqrt{\ }$ " .

(3) $\sqrt{15}$

$\sqrt{15}$ **cannot** be constructed using "Addition method", that is, of the form $\sqrt{15} = \sqrt{a^2 + b^2}$.

The reader may check all limited combinations. (note that $0 < a, b < 4$)
So, if we want to construct $\sqrt{15}$ using only **one** right-angled triangle, we must use "Subtraction method" and we must write $\sqrt{15} = \sqrt{a^2 - b^2}$

Some algebra:
$15 = a^2 - b^2 = (a+b)(a-b)$, where a, b are positive integers.
Since $15 = 15 \times 1 = 5 \times 3$, and a > b, we have two cases:

(i) $\begin{cases} a + b = 15 & ...(1) \\ a - b = 1 & ...(2) \end{cases}$

$\frac{(1)+(2)}{2}$, $a = 8$

$\frac{(1)-(2)}{2}$, $a = 7$

(ii) $\begin{cases} a + b = 5 & ...(3) \\ a - b = 3 & ...(4) \end{cases}$

$\frac{(3)+(4)}{2}$, $a = 4$

$\frac{(3)-(4)}{2}$, $a = 1$

Therefore we have two methods in constructing $\sqrt{15}$:
Using **Figure 3** on Page 1 (not in scale now),

(i) Take AB = 7 and the radius of the circle AE = 8, we get

$$BE = \sqrt{8^2 - 7^2} = \sqrt{64 - 49} = \sqrt{15}$$

(ii) Take AB = 1 and the radius of the circle AE = 4, we get

$$BE = \sqrt{4^2 - 1^2} = \sqrt{16 - 1} = \sqrt{15}$$

(4) $\sqrt{5}$ and $\sqrt{6}$

$\sqrt{5}$ can be constructed by **both** "Addition method" and "Subtraction method" using only **one** right-angled triangle, since $\sqrt{5} = \sqrt{2^2 + 1^2}$ and $\sqrt{5} = \sqrt{3^2 - 2^2}$. Can you build up the algebra behind ?

$\sqrt{6}$ **cannot** be constructed by either "Addition method" or "Subtraction method" using only **one** right-angled triangle . You must use **two** (or more) right-angled triangles as shown in **Figure 2**. Do you know why? How can it be constructed ?

(5) Further and harder investigation

Using only **one** right-angled triangle, what is the condition for n for which we can use "Subtraction method" with to construct $\sqrt{n}$?

Answers:

(4) One possible solution for constructing $\sqrt{6}$ using **two** right-angled triangles :

$$\sqrt{5} = \sqrt{2^2 + 1^2}, \qquad \sqrt{6} = \sqrt{\left(\sqrt{5}\right)^2 + 1^2}$$

(5) Using only **one** right-angled triangle, n must be a product of two odd factors or two even factors in order that $\sqrt{n}$ can be constructed using "Subtraction method" .

For examples,

(a) $\sqrt{7}$ **can** be constructed using "Subtraction method" since

$7 = 7 \times 1$, 7 and 1 are both odd . Also $\sqrt{7} = \sqrt{4^2 - 3^2}$.

(b) $\sqrt{8}$ **can** be constructed using "Subtraction method" since

$8 = 4 \times 2$, 4 and 2 are both even . Also $\sqrt{8} = \sqrt{3^2 - 1^2}$.

(c) $\sqrt{6}$ **cannot** be constructed using "Subtraction method" since

$6 = 6 \times 1 = 3 \times 2$

6 and 1 , 3 and 2 are numbers with one odd and one even.